Fall 2014

1. Since Jan makes a 10% down payment, that leaves 90% left to be paid. 540 paid quarterly means $4 * 540$ paid in a year, so $4 * 540 = .9x$

$\frac{2160}{.9} = x$

$2400 = x$

So the answer is **A. $2400**

2. The lines intersect when $mx + 1 = -mx + 1$.

$mx = -mx$

$2mx = 0$

$x = 0$

$y = m(0) + 1 = 1$

So $x + y = 1$

So the answer is **D. 1**

3. TV's/person = TV's per household/people per household $= \frac{2.5}{3.75} = \frac{2}{3}$

So the answer is **B. 2/3**

4. Adding the average to a set of numbers doesn't change the average, so we can just average the 5 numbers to get our answer.

$\frac{37+49+51+53+75}{5} = 53$

So the answer is **D. 53**

5. $M + m = 9/20$

Since they're perpendicular, $M = -\frac{1}{m}$

Substituting, $-\frac{1}{m} + m = \frac{9}{20}$

$20m^2 - 9m - 20 = 0$

$(4m - 5)(5m + 4) = 0$

So $m = \frac{5}{4}\ or -\frac{4}{5}\ and\ M = the\ other\ one.$

$|M - m| = \left|-\frac{4}{5} - \frac{5}{4}\right| = \left|-\frac{41}{20}\right| = \frac{41}{20}$

So the answer is **D.** $\mathbf{\frac{41}{20}}$

6. Since we are dealing with fractions over 4, we can have $\frac{\frac{x}{4}}{\frac{y}{4}} = \frac{x}{4} - \frac{y}{4}$, with x and y being integers.

   So, $\frac{x}{y} = \frac{x}{4} - \frac{y}{4}$

   $4x = xy - y^2$

   $y^2 = xy - 4x$

   $y^2 = x(y - 4)$

   $\frac{y^2}{y-4} = x$

   Plugging in $y = \frac{x^2}{x-4}$ in the calculator and looking at the table yields 5, 25 as the only odd positive solution, so $x = \frac{25}{4}, y = \frac{5}{4}$

   $x + y = \frac{30}{4} = 7.5$

So the answer is **C. 7.5**

7. Let $x = the\ time\ of\ the\ X10, y = the\ time\ of\ the\ X5, z = the\ time\ of\ the\ X2.$
   $\frac{1}{x} + \frac{1}{y} = \frac{1}{2.4}$
   $\frac{1}{x} + \frac{1}{z} = \frac{1}{3}$
   $\frac{1}{y} + \frac{1}{z} = \frac{1}{4}$
   We need $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.
   Adding all three equations gets us $\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = \frac{1}{2.4} + \frac{1}{3} + \frac{1}{4} = 1$
   So $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$, so it takes 2 hours

So the answer is **E. 2.0**

8. $AMA + TYC = AWAY$.
Since there is a carry and we're only adding two numbers, $A = 1$
Since $A + T + (carry\ from\ M + Y) = 1 + T + (0\ or\ 1) \geq 10, T$ must be 8 or 9
If $T = 9$:
$W = 0$ since 1 is taken, and $M + Y = 1$, since there can't be a carry. So M and Y are 0 and 1, but that isn't possible.
So T must be 8, $W = 0$, and $M + Y = 11$, since C can't be 9 (would lead to Y=0) so there isn't a carry there.
If $M = 2, Y = 9, C = 8$, not possible
If $M = 3, Y = 8$, not possible
If $M = 4, Y = 7, C = 6$, fine
If $M = 5, Y = 6, C = 5$, not possible
If $M = 6, Y = 5, C = 4$, fine
If $M = 7, Y = 4, C = 3$, fine
If $M = 8$, not possible
If $M = 9, Y = 2, C = 1$, not possible
There are 3 possibilities that work

So the answer is **A. 3**

9. Factoring $18x^4 - 11x^2 + 1 = (9x^2 - 1)(2x^2 - 1) =$
$(3x - 1)(3x + 1)(\sqrt{2}x - 1)(\sqrt{2}x + 1)$
So, $= \pm\frac{1}{3}, \pm\frac{1}{\sqrt{2}}$, so there are 2 irrational solutions

So the answer is **C. 2**

10. $p = \frac{r}{3}$
The probability of a green on a button is p. The probability of two greens in two buttons is p^2. The probability of exactly one red is $\binom{2}{1}r^1(1-r)^1 = 2r(1-r)$ and is mutually exclusive with 2 greens. For the probabilities to be equal:
$p = p^2 + 2(r)(1 - r)$
$p = p^2 + 2\left(\frac{p}{3}\right)\left(1 - \frac{p}{3}\right)$
$p = p^2 + \frac{2p}{3} - \frac{2p^2}{9}$
$0 = \frac{7}{9}p^2 - \frac{1}{3}p$
$0 = p(\frac{7}{9}p - \frac{1}{3})$
$\frac{7}{9}p = \frac{1}{3}$
$p = \frac{9}{21} = \frac{3}{7}$
So the answer is **B. 3/7**

11. $f(1) = 3$
$f(x) + f(x) = f(x)f(0) = 2f(x), so\ f(0) = 2$
$f(x) + f(0) = f\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) = \left[f\left(\frac{x}{2}\right)\right]^2$
$f(x) = \left[f\left(\frac{x}{2}\right)\right]^2 - 2$
$f(2) = [f(1)]^2 - 2 = 3^2 - 2 = 7$
$f(3) + f(1) = f(2)f(1) = 21$
$f(3) = 18$
$f(6) = [f(3)]^2 - 2 = 18^2 - 2 = 322$

So the answer is **322**

12. $\sin 15° \,(2 \cos 15°) = \sin 30°$
$\sin 30°(2 \cos 30° = \sin 60°$
This repeats, so the whole thing is equal to $\sin 3840° = \sin 240° = -\frac{\sqrt{3}}{2}$

So the answer is **A.** $-\sqrt{3}/2$

13. $a^6 + b^2 + c^2 = 2014$
$c = \sqrt{2014 - b^2 - a^6}$
Using the calculator, plugging in y=$\sqrt{2014 - 2^2 - X^6}$, looking at the table, and replacing b with prime numbers we eventually get $a = 3, b = 31, c = 18$
$a + b + c = 3 + 31 + 18 = 52$

So the answer is **B. 52**

14. A closed loop by itself splits a plane into 2 regions. Each point hit twice adds an additional region, each point hit three times adds 2 additional regions. So we have $2 + 20 + 2(12) = 46$ regions

So the answer is **D. 46**

15. Let's draw out the possibilities, and for each tell whether it changes if rotated 90°, 180°, or flipped (either vertically or horizontally).

<table>
<tr><th>Shape</th><th>90</th><th>180</th><th>Flip</th><th>Total</th></tr>
<tr><td>==
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<tr><td>Total</td><td></td><td></td><td></td><td>36</td></tr>
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So the answer is **C. 36**

16. This works because 7 divides 21 evenly. Since 13 divides 91 evenly, it'll work for thirteen using 9 times the last digit.

So the answer is **E. 9**

17. We need the distance to (0,0) to be less than the distance to (3,9), (6,6), or (9,3). Using distance equations and squaring both sides, this means:

$x^2 + y^2 \leq (x-3)^2 + (y-9)^2$

$x^2 + y^2 \leq (x-6)^2 + (y-6)^2$

$x^2 + y^2 \leq (x-9)^2 + (y-3)^2$

Working with each one:

$x^2 + y^2 \leq x^2 - 6x + 9 + y^2 - 18y + 81$

$0 \leq -6x - 18y + 90$

$\boldsymbol{y \leq -\frac{1}{3}x + 5}$

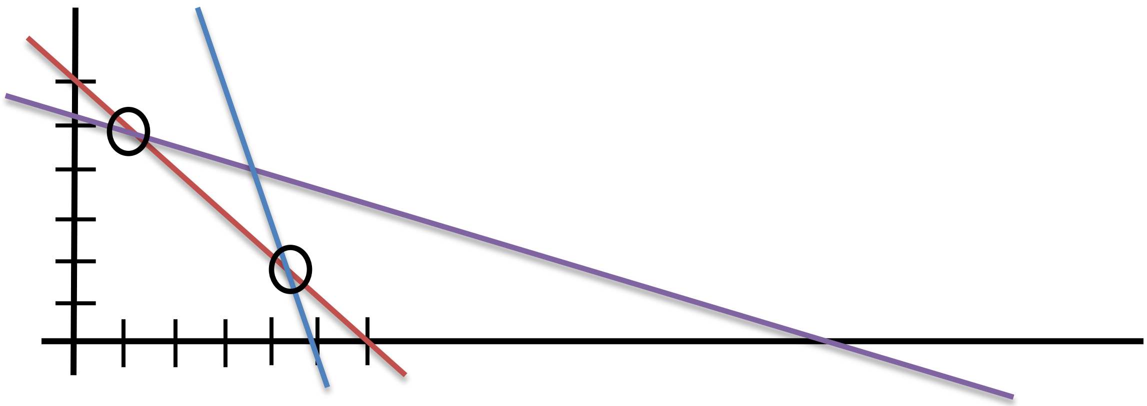
$x^2 + y^2 \leq x^2 - 12x + 36 + y^2 - 12y + 36$

$12y \leq -12x + 72$

$\boldsymbol{y \leq -x + 6}$

$x^2 + y^2 \leq x^2 - 18x + 81 + y^2 - 6y + 9$
$6y \leq -18x + 90$
$\boldsymbol{y \leq -3x + 15}$

Graphing these:

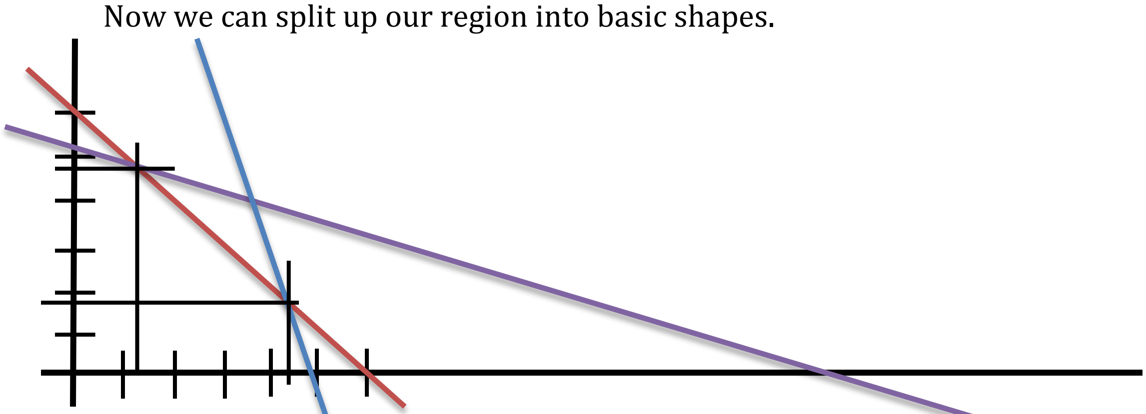


We need to know where the circled points are. One is where $y = -3x + 15$ meets $y = -x + 6$. This shape is symmetric about y=x so the other one will be the inverse.

$-3x + 15 = -x + 6$
$9 = 2x$
$x = \frac{9}{2}$
$y = -\frac{9}{2} + 6 = \frac{3}{2}$

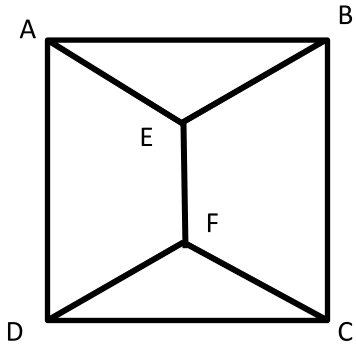
So the points we needed are $(\frac{9}{2}, \frac{3}{2})$ and $(\frac{3}{2}, \frac{9}{2})$.

Now we can split up our region into basic shapes.



We have a $\frac{3}{2} \times \frac{3}{2}$ square (area: 9/4), a 3×3 triangle (area: 9/2), two $\frac{3}{2} \times 3$ rectangles (area: 9/2 x2=9) and two $\frac{3}{2} \times \frac{1}{2}$ triangles (area: 3/8 x2=3/4)
Summing up we have 16.5 square units of area

So the answer is **D 16.5**

18 

$AB = 6$ and the shape is a square, so all of its sides are 6.

$AE = EF = FD = x$

Let's denote the midpoint of AB by M. Then $AM = 3$ and $ME = \frac{1}{2}(6 - x)$.

Using the Pythagorean Theorem, $AE^2 = AM^2 + ME^2$

$x^2 = 3^2 + \left(\frac{6-x}{2}\right)^2$

$x^2 = 9 + 9 - 3x + \frac{x^2}{4}$

$\frac{3}{4}x^2 + 3x - 18 = 0$

$x^2 + 4x - 24 = 0$

$x^2 + 4x + 4 = 28$

$(x + 2)^2 = 28$

$x + 2 = \pm 2\sqrt{7}$

Since it's impossible for x to be negative, $x = 2\sqrt{7} - 2$

So the answer is **B.** $\mathbf{2\sqrt{7} - 2}$

19 $a_8 = a_7 + a_5$

$a_7 = a_6 + a_4$

$a_8 = a_6 + a_4 + a_5$

$a_6 = 30 = a_5 + a_3$

$a_5 = 30 - a_3$

$a_8 = 30 + a_4 + 30 - a_3$

$a_4 = a_3 + 3$

$a_8 = 60 + a_3 + 3 - a_3 = 63$

So the answer is **A. 63**

20. -1 is a zero regardless of what A, B, and C are since plugging in -1 gets $-A + B - C + C - B + A = 0$.

Plugging $\sqrt{3} - 1$ in for x gets us $(44\sqrt{3} - 76)A + (28 - 16\sqrt{3})B + (6\sqrt{3} - 10)C + (4 - 2\sqrt{3})C + (\sqrt{3} - 1)B + A$

$= 44\sqrt{3}A - 15\sqrt{3}B + 4\sqrt{3}C - 75A + 27B - 6C = 0$

Both the integers and the $\sqrt{3}'s$ have to be 0 since A, B, and C are rational, so:

$44A - 15B + 4C = 0$

$-75A + 27B - 6C = 0$

Putting this into the TI-83 as a matrix and using rref on it gets us:

$1\ 0\ \frac{2}{7}$

$0\ 1\ \frac{4}{7}$

So $A = -\frac{2}{7}C, B = -\frac{4}{7}C$.

This gets us $C\left(-\frac{2}{7}x^5 - \frac{4}{7}x^4 + x^3 + x^2 - \frac{4}{7}x - \frac{2}{7}\right) = 0$

Plugging that into y= on the TI-83 and hitting trace, the plugging in the answer choices gets us that $1 - \sqrt{3}$ is not a zero and $\frac{1+\sqrt{3}}{2}$ is a zero.

So the answer is **E. B ($\frac{1+\sqrt{3}}{2}$) and C (-1)**